On A Law of Large Number for Last Passage Percolation on Complete Graph

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- Introduction
- Results
- On the proofs

Last passage percolation on complete graph

$G_n = ([n], E_n)$: the complete graph , where

$$[n] = \{1, 2, \ldots, n\}, E_n = \{\langle i, j\rangle : 1 \le i < j \le n\}.$$

$\{X_e : e \in E_n\}$: i.i.d. positive random variables, edge passage times.

$\Pi_{1,n}$: the set of all self-avoiding paths between vertex 1 and n.

$T(\pi)$: the passage time of path $\pi \in \Pi_{1,n}$

$$T(\pi) = \sum_{e\in\pi} X_e.$$

W_n: the largest passage time among all self-avoiding paths from 1 to n, i.e.,

$$W_n = \sup_{\pi\in\Pi_{1,n}} T(\pi).$$

1. The time constant μ exists!

Theorem 1

For any distribution of edge passage time, the time constant of the model exists. More precisely, there exists some constant $0 < \mu \leq \infty$, *such that*

$$\frac{W_n}{n} \to \mu \quad \text{a.s.}$$

as $n \to \infty$. *In particular, when* $\mu < \infty$, *the above convergence is also in* L_1. *Furthermore,* μ *coincides with the essential supremum of* X_e, *i.e.*

$$\mu = \inf\{x : \mathbb{P}(X_e > x) = 0\}.$$

2. Lower and upper bounds for W_n/n

Let $F(x) = \mathbb{P}(X_e \leq x)$ and $H(x) = 1 - F(x)$, $x \in \mathbb{R}$ be the distribution function and the tail probability function of X_e. By Theorem 1, when $\mu = \infty$, W_n/n tends to ∞ as $n \to \infty$. Here, we give lower and upper bounds to W_n/n as in the following theorem.

Theorem 2

Suppose that $\mu = \infty$. Let f and g be two functions such that $H(f(n)) = \ln n/n$ and $n^2 H(g(n)) \to 0$ as $n \to \infty$. Then, for any $\epsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}\left((1-\epsilon)f(n) \leq \frac{W_n}{n} \leq g(n)\right) = 1.$$

A classification of increasing function

Typically increasing function:

Let's consider the usual exponential function $\exp(x)$. For any $n \in \mathbb{Z}$, write $\exp^n(x)$ as the n-th iteration of the exponential function. Note that, by convention, $\exp^0(x) = x$, $\exp^{-1}(x) = \log(x)$, and $\exp^{-n}(x)$ is the n-th iteration of $\log(x)$ for $n > 0$.

A function ψ is called *typically increasing*, if ψ has the form

$$\psi(x) = a[\exp^n(x)]^b + c$$

with $a, b > 0$, $c \in \mathbb{R}$, $n \in \mathbb{Z}$, for large enough $x \in \mathbb{R}$.

Regularly increasing function:

A strictly increasing function ϕ is called *regularly increasing* if, it is differentiable (at all large enough x), and increases "regularly" compared to typically increasing functions:

for any typically increasing function ψ, IF for some increasing sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = \infty$, one has $\phi(x_n) \geq \psi(x_n)$, THEN, for all large enough x,

- $\phi(x) \geq \psi(x)$;
- $\phi(x) - \psi(x)$ *and* $\psi^{-1}(x) - \phi^{-1}(x)$ *are monotone in* x.

Where ϕ^{-1} and ψ^{-1} are the inverses of ϕ and ψ.

"Regular" means: $\phi - \psi$ and $\psi^{-1} - \phi^{-1}$ do not change their monotonicity frequently!

Assumptions

Rewrite $H(x) = \mathbb{P}(X_e > x)$, the tail probability of X_e, in the form

$$H(x) = e^{-\beta(x)}.$$

We assume $\beta(x)$ satisfy the following technical conditions:

A1: *$\beta(x)$ is second-order differentiable and $\beta''(x)$ is continuous;*

A2: *$\gamma(x) := \beta'(x)$ is monotone. In the case when $\lim_{x\to\infty}\gamma(x) = +\infty$, $\gamma(x)$ increases strictly and the function $s(x)$ solving the equation*

$$\gamma(s(x)) = D\gamma(x), \ \ D > 1,$$

is regularly increasing.

Statement of Result

Theorem 3: Law of Large Number

Suppose that $H(x) = \mathbb{P}(X_e > x) = e^{-\beta(x)}$. If for some $t > 0$,

$$\varphi(t) := \mathbb{E}(e^{tX_e}) < \infty,$$

and the above assumptions A1 and A2 hold, then

$$\frac{W_n}{n\beta^{-1}(\log n)} \to 1$$

in probability as $n \to \infty$. Where β^{-1} is the inverse of β.

For any $x > 0$, let $I(x)$ be the *Legendre transform* of the cumulant generating function $\log \varphi$, namely,

$$I(x) = \sup_{t>0}[xt - \log \varphi(t)].$$

First of all, we have the following lemma.

Lemma 1: Under the condition of Theorem 3, for any $\epsilon > 0$, one has

$$I[(1+\epsilon)\beta^{-1}(\log n)] \geq \log n$$

for all large enough n.

Slowly varying function

A function f is called *slowly varying*, if for any $0 < c < 1$,

$$\lim_{x \to +\infty} \frac{f(cx)}{f(x)} = 1.$$

Let

$$\alpha(x) := \frac{\beta(x)}{x},$$

then by the condition of Theorem 3

$$\lim_{x \to \infty} \alpha(x) = \alpha_0 \in (0, +\infty]$$

and when $\alpha_0 < +\infty$, the function α is slowly varying.

Proof of Lemma 1

We prove Lemma 1 in two steps.

Step 1: The case $\alpha(x) = \dfrac{\beta(x)}{x}$ is slowly varying.

$$\varphi(t) = \int_0^{+\infty} \mathbb{P}(e^{tX_e} > x)dx = 1 + \int_1^{+\infty} H(\frac{\ln x}{t})dx.$$

For large enough x and small $c > 0$, let $\lambda_0 = \alpha(c\beta^{-1}(x))$, one has

$$\begin{aligned} \varphi(t) &\leq e^{c\beta^{-1}(x)t} + \int_{e^{c\beta^{-1}(x)t}}^{+\infty} e^{-\frac{\lambda_0}{t}\log y} dy \\ &= e^{c\beta^{-1}(x)t} \left(\frac{\lambda_0 - \left(1 - e^{-c\beta^{-1}(x)\lambda_0}\right) t}{\lambda_0 - t} \right) \end{aligned}$$

for all $0 < t < \lambda_0$.

Then

$$
\begin{aligned}
& I((1+\epsilon)\beta^{-1}(x)) \\
\geq & \sup_{0<t<\lambda_0} \left[(1+\epsilon-c)\beta^{-1}(x)t - \log\left(\frac{\lambda_0 - \left(1 - e^{-c\beta^{-1}(x)\lambda_0}\right)t}{\log(\lambda_0 - t)} \right) \right] \\
= & \sup_{0<t<\lambda_0} \left[\frac{(1+\epsilon-c)}{\alpha(\beta^{-1}(x))} xt - \log\left(\frac{\lambda_0 - \left(1 - e^{-c\beta^{-1}(x)\lambda_0}\right)t}{\log(\lambda_0 - t)} \right) \right] \\
= & \frac{(1+\epsilon-c)}{\alpha(\beta^{-1}(x))} xt_0 - \log\left[\lambda_0 - \left(1 - e^{-c\beta^{-1}(x)\lambda_0}\right)t_0 \right] + \log(\lambda_0 - t_0).
\end{aligned}
$$

Where $t_0 = t_0(x)$ fits the supremum. It is straightforward to check that

$$\frac{\lambda_0}{t_0(x)} \to 1,$$

$$\frac{1}{x} \log \left[\lambda_0 - \left(1 - e^{-c\beta^{-1}(x)\lambda_0}\right) t_0\right] \to 0 \quad \text{and}$$

$$\frac{1}{x} \log(\lambda_0 - t_0) \to 0$$

as $x \to +\infty$. Thus, for small enough c, we have

$$\frac{(1+\epsilon-c)}{\alpha(\beta^{-1}(x))} t_0 \to (1+\epsilon-c) \frac{\alpha(c\beta^{-1}(x))}{\alpha(\beta^{-1}(x))} > 1$$

for large enough x, then $I((1+\epsilon)\beta^{-1}(x)) \geq x$ for large enough x.

Step 2: The case $\alpha(x) := \dfrac{\beta(x)}{x} \nearrow \infty$ and is NOT slowly varying.

In this step, for any $\epsilon > 0$, we try to find some $t_0 = t_0(n)$, such that

$$(1+\epsilon)\beta^{-1}(\log n)t_0 - \log\varphi(t_0) \geq \log n, \quad \text{for all large } n.$$

Let $\zeta(n) = \gamma(\beta^{-1}(\log n))/\alpha(\beta^{-1}(\log n))$ and let

$$t_0 = \gamma(\beta^{-1}(\log n)) = \zeta(n)\alpha(\beta^{-1}(\log n)).$$

Using the variable substitution $s = \frac{\log x}{\zeta(n)\log n}$, one has

$$\int_1^{+\infty} H(\frac{\log x}{t_0})dx = \zeta(n)\log n \int_0^{+\infty} n^{g(s)}ds,$$

where

$$g(s) = \zeta(n)s - \frac{\beta(s\beta^{-1}(\log n))}{\log n}.$$

Then, for any $s_1 > 0$,

$$\varphi(t_0) = 1 + \zeta(n) \log n \left[\int_0^{s_1} n^{g(s)} ds + \int_{s_1}^{+\infty} n^{g(s)} ds \right]$$

Let s_1 be the solution of equation

$$g(s) = -s$$

such that, for $s > s_1$,

$$g(s) < -s.$$

Noticing that $\beta(x)$ increases strictly, one has $g'(1) = 0$, $g'(s) > 0$ when $s < 1$ and $g'(s) < 0$ when $s > 1$, then

$$g(1) = \sup_{s \geq 0} g(s) = \zeta(n) - 1.$$

So,

$$\begin{aligned}\varphi(t_0) &= 1+\zeta(n)\log n\left[\int_0^{s_1} n^{g(s)}ds + \int_{s_1}^{+\infty} n^{g(s)}ds\right] \\ &\leq 1+\zeta(n)\log n\left[s_1 n^{g(1)} + \int_0^{+\infty} n^{-s}ds\right] \\ &\leq 1+\zeta(n)\log n\cdot\left(s_1 n^{\zeta(n)-1} + constant\right).\end{aligned}$$

Now, for s_1, we claim that,

$$\forall \;\; b>0,\;\; s_1(n)\leq n^b.$$

Then, for any $b>0$,

$$\begin{aligned}\varphi(t_0) &\leq 1+\zeta(n)\log n\cdot\left(s_1 n^{\zeta(n)-1} + constant\right) \\ &\leq \zeta(n)\log n\cdot n^{\zeta(n)-1+b}\end{aligned}$$

for large enough n.

About the claim

Actually, the equation $g(s) = -s$ can be rewritten as

$$\gamma(\beta^{-1}(\log n)) + \alpha(\beta^{-1}(\log n)) = \alpha(s\beta^{-1}(\log n)).$$

Then

$$\gamma(\frac{s_1}{2}\beta^{-1}(\log n)) \leq 4\gamma(\beta^{-1}(\log n)).$$

Let $s(x)$ be the root of the equation

$$\gamma(s(x)) = 4\gamma(x), x > 0. \quad (*)$$

By the following Lemma 2, for any $b > 0$,

$$s(x) \leq e^{bx}$$

for large x. Then,

$$s_1(n) \leq \frac{s_1(n)}{2}\beta^{-1}(\log n) \leq s(\beta^{-1}(\log n)) \leq s(\log n) \leq n^b$$

for large enough n.

Lemma 2 and its proof

Lemma 2: Suppose that $\lim_{x\to+\infty} \alpha(x) = +\infty$ and $\alpha(x)$ is not slowly varying, let $s(x)$ be the solution of equation $(*)$. Then under the condition of Theorem 3, one has

$$\forall \;\; b > 0, \quad s(x) \le e^{bx}$$

for all large x.

Proof:

1. $s(x)$ is regularly increasing (by Assumption A2).
2. If not, α is slowly varying.

Now, we have, for any $b > 0$,

$$\begin{aligned}\varphi(t_0) &= 1 + \zeta(n)\log n \left[\int_0^{s_1} n^{g(s)} ds + \int_{s_1}^{+\infty} n^{g(s)} ds\right] \\ &\leq 1 + \zeta(n)\log n \left[s_1 n^{g(1)} + \int_0^{+\infty} n^{-s} ds\right] \\ &\leq \zeta(n)\log n \cdot n^{\zeta(n)-1+b}\end{aligned}$$

for large enough n.

Hence, we have, for any $b > 0$

$$(1+\epsilon)\beta^{-1}(\log n)t_0 - \log\varphi(t_0) \geq (1+\epsilon)\zeta(n)\log n - \log\varphi(t_0)$$

$$\geq \left[1 + \epsilon\zeta(n) - b - \frac{\log\zeta(n) + \log\log n}{\log n}\right]\log n$$

Noticing that $\zeta(n) \geq 1$, then, by taking $b = \epsilon/2$, we have

$$I((1+\epsilon)\beta^{-1}(\log n)) \geq \left[1 + \frac{1}{2}\epsilon\zeta(n) - \frac{\log\zeta(n) + \log\log n}{\log n}\right]\log n$$

$$\geq \log n$$

for large n. Thus, we finish the second step of the proof.

Proof of Theorem 3

Lower bound part:

Theorem 2 has already given the lower bound

$$f(n) = \beta^{-1}(\log n - \log\log n),$$

so it suffices for us to prove that

$$\lim_{n\to\infty} \frac{\beta^{-1}(\log n - \log\log n)}{\beta^{-1}(\log n)} = 1. \quad (**)$$

When α is *slowly varying*, then for any $0 < c < 1$, one has

$$1 \geq \frac{\beta^{-1}(\log n - \log\log n)}{\beta^{-1}(\log n)} \geq \frac{\beta^{-1}(c\log n)}{\beta^{-1}(\log n)} \to c$$

as $n \to \infty$, $(**)$ follows by taking $c \nearrow 1$.

Otherwise, one has $\alpha_0 = +\infty$ and by A2, $\gamma(x)$ is strictly increasing, hence,

$$\beta^{-1}(\log n) - \beta^{-1}(\log n - \log\log n)$$

$$= \int_{\log n - \log\log n}^{\log n} (\beta^{-1})'(y)dy$$

$$= \int_{\log n - \log\log n}^{\log n} \frac{1}{\gamma(\beta^{-1}(y))}dy$$

$$\leq \frac{\log\log n}{\gamma(\beta^{-1}(\log n - \log\log n))}.$$

Then,

$$\frac{\beta^{-1}(\log n) - \beta^{-1}(\log n - \log\log n)}{\beta^{-1}(\log n)}$$

$$\leq \frac{\log\log n}{\log n - \log\log n} \frac{\alpha(\beta^{-1}(\log n - \log\log n))}{\gamma(\beta^{-1}(\log n - \log\log n))}$$

$$\leq \frac{\log\log n}{\log n - \log\log n} \to 0$$

as $n \to \infty$. Thus we get $(**)$.

Upper bound part:

We have

$$\mathbb{P}(W_n \geq (1+\epsilon)n\beta^{-1}(\log n))$$

$$\leq \sum_{\pi \in \Pi_{1,n}} \mathbb{P}(T(\pi) \geq (1+\epsilon)n\beta^{-1}(\log n)) \qquad (***)$$

$$\leq n! \cdot \mathbb{P}(S_n \geq (1+\epsilon)n\beta^{-1}(\log n)),$$

where $S_n = \sum_{i=1}^{n} X_i$ and $\{X_i\}$ be i.i.d. random variables with the same distribution as X_e.

By Stirling's formula

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + O(\frac{1}{n})\right)$$

and the basic large deviation inequality

$$\mathbb{P}(S_n \geq an) \leq e^{-I(a)n}, \text{ for } a > \mathbb{E}X_e,$$

the upper bound part of the theorem follows from $(***)$ and Lemma 1:

$$\mathbb{P}(W_n \geq (1+\epsilon)n\beta^{-1}(\log n))$$

$$\leq n! \cdot \mathbb{P}(S_n \geq (1+\epsilon)n\beta^{-1}(\log n)),$$

$$\leq n! \cdot e^{-nI((1+\epsilon)\beta^{-1}(\log n))}$$

$$\leq e^{-n}\sqrt{2\pi n}\left(1 + O(\tfrac{1}{n})\right) n^n e^{-n\log n}$$

$$= e^{-n}\sqrt{2\pi n}\left(1 + O(\tfrac{1}{n})\right) \to 0.$$

Thanks for your attention!